NONCOMMUTATIVE GORENSTEIN PROJECTIVE SCHEMES AND GORENSTEIN-INJECTIVE SHEAVES

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Abstract. We prove that if a positively-graded ring R is Gorenstein and the associated torsion functor has finite cohomological dimension, then the corresponding noncommutative projective scheme $\mathrm{Tails}(R)$ is a Gorenstein category in the sense of [10]. Moreover, under this condition, a (right) recollement relating Gorenstein-injective sheaves in $\mathrm{Tails}(R)$ and (graded) Gorenstein-injective R-modules is given.

1. Introduction

1.1. Gorenstein-projective and Gorenstein-injective modules over arbitrary rings are introduced to generalize Auslander's notion of modules of G dimension zero, and later they play central roles in the theory of relative homological algebra and Tate cohomology (see [6, 11, 4]). If the ring is Gorenstein, or more generally, the module category is a Gorenstein category, then these modules behave nicely. Note that via old and new works, those modules appear naturally in the study of singularity categories of rings and schemes ([6, 15, 4, 21, 8, 7]).

Extensions of these notions to Grothendieck categories are considered in [10]. Since a Grothendieck category has enough injective objects but usually not enough projective objects, in these categories we have the notion of Gorenstein-injective objects but perhaps not of Gorenstein-projective objects. The notion of a Gorenstein category is defined in [10]. This is a Grothendieck category with a generator of finite projective dimension, and objects of finite projective dimension coincide with those of finite injective dimension, and where these dimensions are uniformly bounded. Again, when these Gorenstein condition is fulfilled, the Gorenstein-injective objects behave nicely (see [10, Theorem 2.24] and our Proposition 5.2). Two main types of examples of Gorenstien categories are the module category R-Mod over Gorenstein rings R and the category of quasi-coherent sheaves $\mathrm{Qcoh}(\mathbb{X})$ for a locally Gorenstein closed subscheme $\mathbb{X} \subseteq \mathbb{P}^n(k)$ for any commutative noetherian ring k (see [10]). In this paper, we will show that the category of quasi-coherent sheaves on certain noncommutative projective scheme in the sense of [1] is Gorenstein, and under this Gorenstein condition, a (right) recollement on the stable category of Gorenstein-injective sheaves will be given explicitly.

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1.2. We will state our main results in this subsection. Let $R = \oplus_{n\geq 0} R_n$ be a positively-graded ring. Assume that R is left-noetherian, and note that this is equivalent to that R is graded left-noetherian (i.e., any graded left ideal is finitely-generated). Let R-Gr be the category of left $\mathbb{Z}$-graded R-modules with morphisms preserving degrees. Recall the degree-shift functors $(d) : R\text{-Gr} \longrightarrow R\text{-Gr}$ for each integer d: let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded module, define $M(d) = \oplus_{n\in\mathbb{Z}} M(d)_n$ such that $M(d)_n = M_{n+d}$, and the R-module structure is unchanged; (d) acts on morphisms naturally. Then these functors are automorphisms. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded R-module, $m \in M$. We say that m is a *torsion element* if there exists $d \geq 1$ such that $R_{\geq d}.m = 0$, where $R_{\geq d} := \oplus_{n\geq d} R_n$. Denote by $\tau(M) \subseteq M$ the submodule consisting of all the torsion elements, and it is a graded submodule. A module M is said to be a *torsion module* if $\tau(M) = M$. Denote by R-Tor the full subcategory of R-Gr consisting of torsion modules. It is a Serre subcategory, i.e., it is closed under (graded) submodules, quotient modules and extensions. The operation τ induces a functor $\tau : R\text{-Gr} \longrightarrow R\text{-Tor}$, called the *torsion functor*, which is the right adjoint to the inclusion functor $\mathrm{inc} : R\text{-Tor} \longrightarrow R\text{-Gr}$, and in other words, R-Tor is a localizing Serre subcategory. By [1, Proposition 2.2(2)], the subcategory R-Tor is closed under essential monomorphisms, that is, given an essential monomorphism $M \longrightarrow M'$ in R-Gr, if M is a torsion module so is M'. In summary, our Setup 4.2 in section 4 applies in this situation.

Use the above notation. Artin and Zhang define the following quotient abelian category in the sense of Gabriel

$$\mathrm{Tails}(R) \ := \ R\text{-Gr}/R\text{-Tor}.$$

We denote by $\pi : R\text{-Gr} \longrightarrow \mathrm{Tails}(R)$ the canonical functor, and set $\mathcal{O} = \pi(R)$ (see [1] for details). In this case, π admits a right adjoint which will be denoted by $\omega : \mathrm{Tails}(R) \longrightarrow R\text{-Gr}$. Since the subcategory R-Tor is closed under the degree-shift functors, thus we get the induced functors $(d) : \mathrm{Tails}(R) \longrightarrow \mathrm{Tails}(R)$ for each d, which are still automorphisms. The category $\mathrm{Tails}(R)$ is sometimes called the *category of tails* of R, or it is known as the noncommutative projective scheme: more precisely, write $\mathbb{X} = (\mathrm{Tails(R)}, \mathcal{O}, (1))$, then we call the triple $\mathbb{X}$ the *(polarized) general projective scheme*, and $\mathcal{O}$ the *structure sheaf*, (1) the *twist functor*, we even write $\mathrm{Qcoh}(\mathbb{X}) = \mathrm{Tails}(R)$, which is referred as the category of *quasi-coherent sheaves* on $\mathbb{X}$, and the graded ring R is called the *homogeneous coordinate ring* of $\mathbb{X}$.

We will be concerned with the question of when the category $\mathrm{Tails}(R)$ is Gorenstein. A natural condition is that the ring R is Gorenstein: recall that a ring R is said to be *Gorenstein* if R is two-sided noetherian and R has finite injective dimension both as the left and right regular module. Similarly one defines the notion of *graded Gorenstein rings*. Thanks to a remark by Van den Bergh ([27, p.670, line 9-11]), we infer that a graded module M has finite injective dimension in R-Gr if and only if M has finite injective dimension as an ungraded module. Thus the graded ring R is graded Gorenstein if and only if it is Gorenstein as an ungraded ring; and in this case, one sees easily that R-Gr is a Gorenstein category (by comparing or applying the results in [11, section 9.1]). Another condition we want to impose is that the functor τ has finite cohomological dimension, that is, there exists $d \geq 1$ such that the n-th right derived functor $R^n\tau = 0$ for $n \geq d$. The condition is satisfied in commutative algebraic geometry, more precisely, if the graded noetherian ring R is commutative, generated by R_0 and R_1, and R_0 is an affine algebra over some field,

then the functor τ always has finite cohomological dimension (to see this well-known fact, note that R has finite Krull dimension, first apply the famous Serre theorem ([17, p.125, Ex. 5.9]) and then Grothendieck's vanishing theorem ([17, p.208]) to obtain that the structure sheaf $\mathcal{O}$ has finite projective dimension, and then use our Lemma 4.3). More generally, for any noetherian commutative graded ring R the torsion functor τ has finite cohomological dimension. To see this, first note that τ coincides with the local cohomology functor with respect to the ideal $R_{\geq 1}$ (compare [27, section 4]) and then we can compute local cohomology via the Koszul complex of a finite set of generators of the ideal $R_{\geq 1}$ ([13, Theorem 2.3 and Theorem 2.8]).

We have our first result, which is a direct consequence of Corollary 4.4. One may compare it with the main example of [10, Theorem 3.8] in commutative algebraic geometry.

Theorem A *Let $R = \oplus_{n\geq 0} R_n$ be a graded Gorenstein ring such that the associated torsion functor τ has finite cohomological dimension. Then the category of tails* $\mathrm{Tails}(R)$ *is a Gorenstein category.*

Next we will be concerned with the full subcategory of Gorenstein-injective objects (or sheaves) in $\mathrm{Tails}(R)$ and more precisely its stable category modulo injective objects. Recall that in an abelian category $\mathcal{A}$ with enough injective objects, an object G is said to be Gorenstein-injective ([11]), if there exists an exact complex $I^\bullet$ of injective objects such that the Hom complex $\mathrm{Hom}_{\mathcal{A}}(Q, I^\bullet)$ is still exact for any injective object Q, and $G = Z^0(I^\bullet)$ is the 0-th cocycle. The full subcategory of Gorenstein-injective objects will be denoted by $\mathrm{GInj}(\mathcal{A})$. Note that $\mathrm{GInj}(\mathcal{A})$ is a Frobenius exact category and its relative injective-projective objects are precisely the injective objects in $\mathcal{A}$ (see our Lemma 5.1). Then the stable category $\underline{\mathrm{GInj}}(\mathcal{A})$ modulo injective objects has a canonical triangulated structure by [14, Chapter 1, section 2].

We have our second result which relates the category of Gorenstein-injective objects (or sheaves) in $\mathrm{Tails}(R)$ to the category of Gorenstein-injective R-modules. It is a direct consequence of Theorem 5.3.

Theorem B *Use the same assumption in Theorem A. Then we have a right recollement of triangulated categories:*

$$\underline{\mathrm{GInj}}(R\text{-Tor}) \underset{\underline{\tau}}{\overset{\mathrm{inc}}{\rightleftarrows}} \underline{\mathrm{GInj}}(R\text{-Gr}) \underset{\underline{\omega}}{\overset{\underline{\pi}}{\rightleftarrows}} \underline{\mathrm{GInj}}(\mathrm{Tails}(R)),$$

where the functors are induced from the ones between the abelian categories.

Let us remark that we actually obtain a recollement in this situation. By Theorem A and Proposition 5.2, we observe that an acyclic (= exact) complex of injective objects in $\mathrm{Tails}(R)$ is totally-acyclic, and then by [21, Proposition 7.2], we infer that $\underline{\mathrm{GInj}}(\mathrm{Tails}(R))$ is triangle equivalent to the *stable derived category* $S(\mathrm{Tails}(R))$ in the sense of Krause ([21, section 5]). In particular, by noting that the Grothendieck category $\mathrm{Tails}(R)$ is locally noetherian and then by [21, Corollary 5.4], the stable category $\underline{\mathrm{GInj}}(\mathrm{Tails}(R))$ is compactly generated. Recall a well-known natural isomorphism (see [1, equation (3.12.2)]), for any sheaf $\mathcal{M} \in \mathrm{Tails}(R)$, $\omega(\mathcal{M}) \simeq \oplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Tails}(R)}(\mathcal{O}, \mathcal{M}(d))$, and since the structure sheaf $\mathcal{O}$ is a noetherian object, we infer that the functor ω preserves arbitrary coproducts, and consequently, the

induced triangle functor $\underline{\omega} : \underline{\mathrm{GInj}}(\mathrm{Tails}(R)) \longrightarrow \underline{\mathrm{GInj}}(R\text{-Gr})$ also preserves arbitrary coproducts. Applying Brown representability theorem ([24, Theorem 8.4.4]) to the functor $\underline{\omega}$, we obtain a right adjoint functor π' for $\underline{\omega}$, and thus consequently, we obtain also a right adjoint functor i' for $\underline{\tau}$, and both π' and i' are triangle functors (see the arguments in **2.2**). In summary, we obtain a recollement in this situation, expressed in the following diagram

$$\begin{tikzcd} \underline{\mathrm{GInj}}(\mathrm{Tails}(R)) \arrow[r, "\underline{\omega}"] & \underline{\mathrm{GInj}}(R\text{-Gr}) \arrow[l, shift right=2ex, "\pi"'] \arrow[l, shift left=2ex, "\pi'"] \arrow[r, "\underline{\tau}"] & \underline{\mathrm{GInj}}(R\text{-Tor}). \arrow[l, shift right=2ex, "\mathrm{inc}"'] \arrow[l, shift left=2ex, "i'"] \end{tikzcd}$$

1.3. The paper is organized as follows : in section 2, we review Gabriel's theory of quotient abelian categories and Verdier's theory of quotient triangulated categories, and their right recollements; in section 3, we recall the definition of Gorenstein categories in [10] with slight modifications and we characterize the Gorenstein condition by the bounded derived category; in section 4, we consider when the quotient abelian category of a Gorenstein category is still Gorenstein, and we obtain our main results Theorem 4.1 and Corollary 4.4, which give our Theorem A; in the final section, we consider the stable category of Gorenstein-injective objects in abelian categories, and we obtain a right recollement relating the stable categories of Gorenstein-injective objects in the abelian category and its quotient category, see Theorem 5.3, which gives our Theorem B.

2. Preliminaries

2.1. Let us recall Gabriel's theory of quotient abelian categories. Let $\mathcal{A}$ be an arbitrary abelian category. Let $\mathcal{B} \subseteq \mathcal{A}$ be a *Serre subcategory*, that is, $\mathcal{B} \subseteq \mathcal{A}$ is a full subcategory which is closed under subobjects, quotient objects and extensions, or equivalently, for any short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{A}$, Y lies in $\mathcal{B}$ if and only if X and Z do. Recall the *quotient category* $\mathcal{A}/\mathcal{B}$ is defined by taking the inverse, formally, of the morphisms $f : X \longrightarrow Y$ in $\mathcal{A}$ such that both Ker f and Coker f lie in $\mathcal{B}$. More precisely, denote by Σ the class of such morphisms, which is a multiplicative system on $\mathcal{A}$, then the category $\mathcal{A}/\mathcal{B}$ is given as follows: its objects coincide with the ones in $\mathcal{A}$; and morphisms are given by (right) fractions (= roofs) $f/s : X \overset{s}{\Longleftarrow} Z \overset{f}{\longrightarrow} Y$ modulo certain equivalence relation, where $s \in \Sigma$; composition of fractions are given in [24, Appendix A.2]. Then we have a canonical functor $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$. Note that π sends morphism $f : X \longrightarrow Y$ to the trivial fraction $f/\mathrm{Id}_X : X \overset{\mathrm{Id}_X}{\Longleftarrow} X \overset{f}{\longrightarrow} Y$. Let us remark that the morphism space in $\mathcal{A}/\mathcal{B}$ can also be expressed as certain colimits of morphism spaces in $\mathcal{A}$ as follows

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(\pi(X), \pi(Y)) \;=\; \mathrm{colimt}_{X', Y'}\, \mathrm{Hom}_{\mathcal{A}}(X', Y/Y'),$$

where X' and Y' run over all the subobjects of X and Y such that $X/X' \in \mathcal{B}$ and $Y' \in \mathcal{B}$.

We collect some needed properties of the quotient category: the category $\mathcal{A}/\mathcal{B}$ is abelian and π is an exact functor with Ker $\pi = \mathcal{B}$, or in other words, there is a short "exact sequence" of abelian categories:

$$\mathcal{B} \overset{\mathrm{inc}}{\longrightarrow} \mathcal{A} \overset{\pi}{\longrightarrow} \mathcal{A}/\mathcal{B}$$

where inc is the inclusion functor; moreover, any short exact sequence in $\mathcal{A}/\mathcal{B}$ is isomorphic to the image of some short exact sequence in $\mathcal{A}$.

Later we will be mainly interested in the localizing Serre subcategories. Recall that a Serre subcategory $\mathcal{B} \subseteq \mathcal{A}$ is said to be *localizing* provided that the inclusion functor $\mathrm{inc} : \mathcal{B} \hookrightarrow \mathcal{A}$ admits a right adjoint. This adjoint functor will be denoted by τ and called the *torsion functor*. In this case, the quotient functor $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$ also admits a right adjoint, denoted by $\omega : \mathcal{A}/\mathcal{B} \longrightarrow \mathcal{A}$. Observe that both τ and ω are fully-faithful functors. Consider the full subcategory $\mathcal{B}^{\perp} := \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, X) = \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{B}, X) = 0\}$, whose objects are said to be $\mathcal{B}$-*local* ([24, Appendix A.2]). In this case $\mathcal{B}^{\perp}$ coincides with the essential image $\mathrm{Im}\ \omega$ of the functor ω. A remarkable fact is that the full subcategory $\mathcal{B}^{\perp}$ is a Giraud subcategory of $\mathcal{A}$. Recall that a full subcategory $\mathcal{B}'$ is said to be a *Giraud subcategory* [26] if the inclusion functor $\mathcal{B}' \hookrightarrow \mathcal{A}$ admits a left adjoint functor $\sigma : \mathcal{A} \longrightarrow \mathcal{B}'$ which preserves kernels. Conversely, given a Giraud subcategory $\mathcal{B}'$, the kernel $\mathrm{Ker}\ \sigma$ which coincides with ${}^{\perp}\mathcal{B}' := \{X \in \mathcal{B} \mid \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{B}') = 0\}$ is a localizing Serre subcategory. In fact, this gives a bijection between the class of Serre subcategories and the class of Giraud subcategories.

Another related concept is the right recollement of abelian categories. Let $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be abelian categories. A *right recollement* of $\mathcal{A}$ with respect to $\mathcal{A}'$ and $\mathcal{A}''$ is expressed by the following diagram of functors

$$\begin{tikzcd} \mathcal{A}' \arrow[r, shift left, "i_!"] & \mathcal{A} \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] & \mathcal{A}'' \arrow[l, shift left, "j_*"] \end{tikzcd}$$

satisfying the following conditions

(1). The pairs $(i_!, i^!)$ and (j^*, j_*) are adjoint pairs.
(2). The functors $i_!$ and j^* are exact such that $j^* i_! = 0$.
(3). The functors $i_!$ and j_* are full embeddings.
(4). For each $X \in \mathcal{A}$, there is an exact sequence

$$0 \longrightarrow i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(A),$$

where the two morphisms are the adjunction morphisms of the corresponding adjoint pairs. Compare [25] and [12, 4.1].

In the right recollment, one can show that $\mathrm{Ker}\ j^* = \mathrm{Im}\ i_!$ and thus $\mathrm{Im}\ i_!$ is a Serre subcategory, and via the functor j^*, there exists a natural equivalence of abelian categories $\mathcal{A}/\mathrm{Im}\ i_! \simeq \mathcal{A}''$. Moreover since the pair $(i_!, i^!)$ is adjoint, we infer that $\mathrm{Im}\ i_!$ is a localizing Serre subcategory. Conversely, given a localizing Serre subcategory $\mathcal{B}$, one has a natural exact sequence $0 \longrightarrow \tau(X) \longrightarrow X \longrightarrow \omega(\pi(X))$ for each $X \in \mathcal{A}$, and note that the two morphisms involved are the adjunction morphisms of the corresponding adjoint pairs, therefore we have a right recollement of abelian categories:

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift left, "\mathrm{inc}"] & \mathcal{A} \arrow[l, shift left, "\tau"] \arrow[r, shift left, "\pi"] & \mathcal{A}/\mathcal{B}. \arrow[l, shift left, "\omega"] \end{tikzcd}$$

Observe that, in a proper sense, every right recollement of abelian categories is given in this way.

2.2. Verdier's theory of quotient triangulated categories is analogous to Gabriel's theory. Let $\mathcal{C}$ be a triangulated category whose shift functor is denoted by [1]. Let

$\mathcal{D} \subseteq \mathcal{C}$ be a thick triangulated subcategory. Set Σ to be the class of morphisms $X \xrightarrow{f} Y$ such that it fits in a triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$ with $Z \in \mathcal{D}$. This is a multiplicative system on $\mathcal{C}$ which is compatible with the triangulation. Recall that the *quotient category* $\mathcal{C}/\mathcal{D}$ is the localization of $\mathcal{C}$ with respect to Σ, and as in the case of abelian categories, morphisms in $\mathcal{C}/\mathcal{D}$ are expressed as (right) fractions. Note that the quotient category $\mathcal{C}/\mathcal{D}$ has a unique triangulation such that its triangles are precisely those isomorphic to the image of some triangles in $\mathcal{C}$, and thus the quotient functor $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{D}$ is a triangle functor. Observe that $\mathrm{Ker}\ \pi = \mathcal{D}$. We say that there is a short "exact sequence" of triangulated categories

$$\mathcal{D} \xrightarrow{\mathrm{inc}} \mathcal{C} \xrightarrow{\pi} \mathcal{C}/\mathcal{D}$$

A triangulated analogue of localizing Serre subcategory is a right admissible subcategory. Recall that a thick triangulated subcategory $\mathcal{D} \subseteq \mathcal{C}$ is said to be *right admissible* (or *a Bousfield subcategory*) provided that the inclusion functor $\mathrm{inc} : \mathcal{D} \hookrightarrow \mathcal{C}$ admits a right adjoint (see [24, Chapter 9] and compare [5]). The adjoint will be denoted also by τ, and thanks to Keller ([19, 6.6 and 6.7]), the functor τ is a triangle functor such that the adjunction morphisms are natural morphisms of triangle functors. In this case, the quotient functor π also admits a right adjoint functor $\omega : \mathcal{C}/\mathcal{D} \longrightarrow \mathcal{C}$. Observe that both τ and ω are fully-faithful. Consider the full subcategory $\mathcal{D}^{\perp} = \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(\mathcal{D}, X) = 0\}$, and one observes that this is a thick triangulated subcategory. It is remarkable that the essential image $\mathrm{Im}\ \omega$ of ω coincides with $\mathcal{D}^{\perp}$ and thus the triangulated subcategory $\mathcal{D}^{\perp}$ is left admissible. In fact, this gives a bijection between the class of right admissible subcategories and the class of left admissible subcategories.

Another related concept is the right recollement of triangulated categories. Let $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ be triangulated categories. Recall that a *right recollement* of $\mathcal{C}$ with respect to $\mathcal{C}'$ and $\mathcal{C}''$ is expressed by the following diagram of triangle functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, shift left, "i_!"] & \mathcal{C} \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] & \mathcal{C}'' \arrow[l, shift left, "j_*"] \end{tikzcd}$$

satisfying the following conditions
(1). The pairs $(i_!, i^!)$ and (j^*, j_*) are adjoint pairs.
(2). $j^* i_! = 0$.
(3). The functors $i_!$ and j_* are full embeddings.
(4). For each $X \in \mathcal{C}$, there is a triangle

$$i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow (i_! i^!(X))[1]$$

where the left two morphisms are the adjunction morphisms of the corresponding adjoint pairs. For details, see [3, 25] and [21, section 3].

In a right recollement, one has $\mathrm{Ker}\ j^* = \mathrm{Im}\ i_!$ and thus $\mathrm{Im}\ i_!$ is a thick triangulated subcategory, and via the functor j^*, there exists a natural equivalence of triangulated categories $\mathcal{C}/\mathrm{Im}\ i_! \simeq \mathcal{C}''$. Moreover since the pair $(i_!, i^!)$ is adjoint, we infer that $\mathrm{Im}\ i_!$ is right admissible. Conversely, given a right admissible subcategory $\mathcal{D}$, one has a natural triangle $\tau(X) \longrightarrow X \longrightarrow \omega(\pi(X)) \longrightarrow \tau(X)[1]$ for each $X \in \mathcal{C}$, and note that the left two morphisms are the adjunction morphisms of the corresponding adjoint

pairs, therefore we have a right recollement of triangulated categories:

$$\begin{tikzcd} \mathcal{D} \arrow[r, shift left, "\mathrm{inc}"] & \mathcal{C} \arrow[l, shift left, "\tau"] \arrow[r, shift left, "\pi"] & \mathcal{C}/\mathcal{D}. \arrow[l, shift left, "\omega"] \end{tikzcd}$$

Note that every right recollement of triangulated categories is (in a certain sense) given this way.

Let us end this section by noting that under some conditions a localizing Serre subcatgory of an abelian category may induce a right admissible subcategory of the derived category, or equivalently, a right recollement of derived categories. For an abelian category $\mathcal{A}$, denote by $D^b(\mathcal{A})$ (resp. $D^+(\mathcal{A})$) the derived category of bounded complexes (resp. bounded-below complexes) on $\mathcal{A}$. A full subcategory $\mathcal{B} \subseteq \mathcal{A}$ is said to *have enough $\mathcal{A}$-injective objects* provided that for each object $X \in \mathcal{B}$, there is a monomorphism $X \hookrightarrow I$ in $\mathcal{A}$ where $I \in \mathcal{B}$ is an injective object in $\mathcal{A}$.

Lemma 2.1. *Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a localizing Serre subcategory. Assume that both $\mathcal{A}$ and $\mathcal{A}/\mathcal{B}$ have enough injective objects, and $\mathcal{B}$ has enough $\mathcal{A}$-injective objects. Suppose the corresponding right recollement is given as in* **2.1**. *Then we have a right recollement of derived ctegories:*

$$\begin{tikzcd} D^+(\mathcal{B}) \arrow[r, shift left, "\mathrm{inc}"] & D^+(\mathcal{A}) \arrow[l, shift left, "R^+\tau"] \arrow[r, shift left, "\pi"] & D^+(\mathcal{A}/\mathcal{B}), \arrow[l, shift left, "R^+\omega"] \end{tikzcd}$$

where $R^+\tau$ and $R^+\omega$ are the derived functors, and here inc *and π are applied on complexes term by term.*

Proof. By [16, Chapter I, Proposition 4.8], the natural functor $D^+(\mathcal{B}) \longrightarrow D^+_{\mathcal{B}}(\mathcal{A})$ is a triangle equivalence, where $D^+_{\mathcal{B}}(A)$ denotes the full triangulated subcategory of $D^+(\mathcal{A})$ consisting of complexes with all the cohomologies lying in $\mathcal{B}$. By the dual of [9, Lemma (1.1)], or the proof of [22, Corollary 3.3], we infer that $(\mathrm{inc}, R^+\tau)$ and $(\pi, R^+\omega)$ are adjoint pairs. Now our recollement follows directly from Miyachi's "exact sequence" of derived categories $D^+_{\mathcal{B}}(\mathcal{A}) \xrightarrow{\mathrm{inc}} D^+(\mathcal{A}) \xrightarrow{\pi} D^+(\mathcal{A}/\mathcal{B})$ (see [22, Theorem 3.2]). ■

3. Gorenstein Categories

In this section, we will recall the definition of Gorenstein categories and their basic properties. Before that, we will study the cohomological dimensions of complexes for later use.

3.1. Let us recall the notion of cohomological dimensions in an abelian category. Let $\mathcal{A}$ be an abelian category and assume that the bounded derived category $D^b(\mathcal{A})$ is defined. We will always identify $\mathcal{A}$ as the full subcategory of $D^b(\mathcal{A})$ consisting of stalk complexes concentrated at degree 0. In particular, the n-th extension group of two objects X, Y of $\mathcal{A}$ is defined to be $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) := \mathrm{Hom}_{D^b(\mathcal{A})}(X, Y[n])$ for any $n \in \mathbb{Z}$, where $[n]$ denotes the n-th power of the shift functor ([16, p.62]). Then we have $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0$ if $n < 0$, and $\mathrm{Ext}^0_{\mathcal{A}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(X, Y)$. Recall that the *projective dimension* of an object X is defined by proj.dim $X := \sup \{n \geq 0 \mid$ there exists $Y \in \mathcal{A}$, $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \neq 0\}$. Dually one defines the *injective dimension* of Y, denoted by inj.dim Y. The *global dimension* of the category $\mathcal{A}$ is defined to be gl.dim $\mathcal{A} := \sup \{\text{proj.dim } X \mid X \in \mathcal{A}\}$. Observe that gl.dim $\mathcal{A} = \sup \{\text{inj.dim } Y \mid Y \in \mathcal{A}\}$. Also we need the finitistic dimensions: the *finitistic projective dimension* of $\mathcal{A}$ is defined

to be fin.pd $\mathcal{A} := \sup\{\text{proj.dim } X < \infty \mid X \in \mathcal{A}\}$; similarly, we define the *finitistic injective dimension* fin.id $\mathcal{A}$. Observe that fin.pd $\mathcal{A}$, fin.id $\mathcal{A} \leq$ gl.dim $\mathcal{A}$, and if gl.dim $\mathcal{A} < \infty$, then we have fin.pd $\mathcal{A}$ = gl.dim $\mathcal{A}$ = fin.id $\mathcal{A}$.

For later use, we generalize the above and define the projective and injective dimensions for bounded complexes. Let $X^\bullet \in D^b(\mathcal{A})$ be a bounded complex. Define the *projective dimension* of $X^\bullet$ as

$$\text{proj.dim } X^\bullet := \sup\{n \in \mathbb{Z} \mid \text{ there exists } Y \in \mathcal{A}, \text{Hom}_{D^b(\mathcal{A})}(X^\bullet, Y[n]) \neq 0\}.$$

Dually, the *injective dimension* is

$$\text{inj.dim } Y^\bullet := \sup\{n \in \mathbb{Z} \mid \text{ there exists } X \in \mathcal{A}, \text{Hom}_{D^b(\mathcal{A})}(X[-n], Y^\bullet) \neq 0\}.$$

If $X^\bullet = X$ is a stalk complex concentrated at degree 0, then its projective dimension coincides with the one given in the preceding paragraph. Similar remark holds for $Y^\bullet$. Note that two quasi-isomorphic complexes necessarily have the same projective and injective dimensions.

Lemma 3.1. *Let $X^\bullet = (X^n, d^n)_{n\in\mathbb{Z}}$ be a bounded complex. Then we have* proj.dim $X^\bullet \leq \max\{\text{proj.dim } X^n - n \mid X^n \neq 0\}$ *and* inj.dim $X^\bullet \leq \max\{\text{inj.dim } X^n + n \mid X^n \neq 0\}$.

Proof. First note two facts: for each complex $X^\bullet$ and each $n \in \mathbb{Z}$, proj.dim $X^\bullet[n] =$ proj.dim $X^\bullet + n$; for each triangle $X'^\bullet \longrightarrow X^\bullet \longrightarrow X''^\bullet \longrightarrow X'^\bullet[1]$ in $D^b(\mathcal{A})$, applying the cohomological functor $\text{Hom}_{D^b(\mathcal{A})}(-, Y[n])$ for any $Y \in \mathcal{A}$ to it, we obtain that $\text{Hom}_{D^b(\mathcal{A})}(X^\bullet, Y[n]) \neq 0$ implies that $\text{Hom}_{D^b(\mathcal{A})}(X'^\bullet, Y[n]) \neq 0$ or $\text{Hom}_{D^b(\mathcal{A})}(X''^\bullet, Y[n]) \neq 0$, therefore we have proj.dim $X^\bullet \leq \max\{\text{proj.dim } X'^\bullet, \text{proj.dim } X''^\bullet\}$. Note that a given bounded complex $X^\bullet$ is an iterated extension of the stalk complexes $X^n[-n]$, and thus applying the above two facts repeatedly, we prove the first inequality, and similarly the second one. ■

Following [23, Appendix], we denote by $D^b(\mathcal{A})_{\text{fpd}}$ (resp. $D^b(\mathcal{A})_{\text{fid}}$) the full subcategory of $D^b(\mathcal{A})$ consisting of complexes of finite projective dimension (resp. finite injective dimension). Observe that they are thick triangulated subcategories. Note that if $\mathcal{A}$ has enough projective objects (resp. injective objects), the complexes in $D^b(\mathcal{A})_{\text{fpd}}$ (resp. $D^b(\mathcal{A})_{\text{fid}}$) have a characterization by resolutions. For example, we recall that the abelian category $\mathcal{A}$ is said to *have enough injective objects* if every object can be embedded into an injective object. Then the following is well-known (compare [16, Chapter I, Proposition 7.6]).

Lemma 3.2. *Let $\mathcal{A}$ be an abelian category with enough injective objects, let $X^\bullet = (X^n, d^n) \in D^b(\mathcal{A})$, $n_0 \in \mathbb{Z}$. The following are equivalent:*
(1). inj.dim $X^\bullet \leq n_0$.
(2). For any quasi-isomorphism $X^\bullet \longrightarrow I^\bullet$ with $I^\bullet = (I^n, \partial^n)$ a bounded-below complex of injective objects, then Ker ∂^{n_0} *is injective and $H^n(I^\bullet) = 0$ for $n > n_0$.*
(3). There exists a quasi-isomorphism $X^\bullet \longrightarrow I^\bullet$ where $I^\bullet$ is a bounded complex of injective objects with $I^n = 0$ for $n > n_0$.

Consequently, we have a natural equivalence of triangulated categories $K^b(\mathcal{I}) \simeq D^b(\mathcal{A})_{\text{fid}}$, where $\mathcal{I}$ is the full subcategory consisting of all the injective objects and $K^b(\mathcal{I})$ its bounded homotopy category.

Proof. For "(1) $\Longrightarrow$ (2)", first note that $H^n(X^\bullet) = 0$ for $n > n_0$. Otherwise, the natural chain map Ker $d^n[-n] \longrightarrow X^\bullet$ induces an epimorphism on the n-th cohomology groups, and thus it is not zero in $D^b(\mathcal{A})$. However this is impossible

since inj.dim $X^\bullet < n$. Hence via the quasi-isomorphism we have $H^n(I^\bullet) = 0$ for $n > n_0$. Therefore $X^\bullet$ is isomorphic to the good truncated complex $\tau^{\leq n_0} I^\bullet = \cdots \longrightarrow I^{n_0-2} \longrightarrow I^{n_0-1} \longrightarrow \mathrm{Ker}\ \partial^{n_0} \longrightarrow 0$ in the derived category $D^b(\mathcal{A})$. Note that we have a natural triangle in $D^b(\mathcal{A})$

$$(\sigma^{\leq n_0-1} I^\bullet)[-1] \longrightarrow \mathrm{Ker}\ \partial^{n_0}[-n_0] \longrightarrow \tau^{\leq n_0} I^\bullet \longrightarrow \sigma^{\leq n_0-1} I^\bullet$$

where $\sigma^{\leq n_0-1} I^\bullet = \cdots \longrightarrow I^{n_0-2} \longrightarrow I^{n_0-1} \longrightarrow 0$ is the brutal truncated complex. Thus since inj.dim $\tau^{\leq n_0} I^\bullet =$ inj.dim $X^\bullet \leq n_0$ and as we will easily see below in the proof of "(3) $\Longrightarrow$ (1)" that inj.dim $\sigma^{\leq n_0-1} I^\bullet \leq n_0 - 1$, and thus from the triangle above, we infer that inj.dim Ker $\partial^{n_0}[-n_0] \leq n_0$, and thus since inj.dim Ker $\partial^{n_0}[-n_0] =$ inj.dim Ker $\partial^{n_0} + n_0$, we infer that inj.dim Ker $\partial^{n_0} \leq 0$, and then it necessarily forces that inj.dim Ker $\partial^{n_0} = 0$, that is, Ker ∂^{n_0} is injective.

For "(2) $\Longrightarrow$ (3)", first take a quasi-isomorphism $X^\bullet \longrightarrow I^\bullet$ where $I^\bullet = (I^n, \partial^n)$ is a bounded-below complex of injective objects ([16, p.42]). Then by (2), the subcomplex $\tau^{\leq n_0} I^\bullet = \cdots \longrightarrow I^{n_0-2} \longrightarrow I^{n_0-1} \longrightarrow \mathrm{Ker}\ \partial^{n_0} \longrightarrow 0$ is a bounded complex of injective objects, and moreover the natural chain map $\tau^{\leq n_0} I^\bullet \longrightarrow I^\bullet$ is split-mono in the category of complexes. Take its retraction to be $s^\bullet$. Then the composite $X^\bullet \longrightarrow I^\bullet \xrightarrow{s^\bullet} \tau^{\leq n_0} I^\bullet$ fulfills (3).

The implication "(3) $\Longrightarrow$ (1)" is immediate, since we have inj.dim $X^\bullet =$ inj.dim $I^\bullet$ and the canonical isomorphism $\mathrm{Hom}_{D^b(\mathcal{A})}(X[-n], I^\bullet) \simeq \mathrm{Hom}_{K^b(\mathcal{A})}(X[-n], I^\bullet)$ for any $X \in \mathcal{A}$, here $K^b(\mathcal{A})$ is the bounded homotopy category of $\mathcal{A}$. ∎

3.2. We will recall the definition of Gorenstein categories. Recall that an abelian category $\mathcal{A}$ is said to *satisfy the* (AB4) *condition* if it has arbitrary (set-indexed) coproducts and coproducts preserve short exact sequences. An object $T \in \mathcal{A}$ is said to be *a generator* if the functor $\mathrm{Hom}_{\mathcal{A}}(T, -) : \mathcal{A} \longrightarrow \mathrm{Ab}$ is faithful, where Ab denotes the category of abelian groups. Observe that if the abelian category $\mathcal{A}$ has arbitrary coproducts, then an object T is a generator if and only if any object in $\mathcal{A}$ is a quotient of a coproduct of copies of T.

This definition is essentially given by Enochs, Estrada, García Rozas ([10, Definition 2.18]). One may note that there are other different but related notions of Gorenstein categories, see [4, section 4] and [20, 3.2].

Definition 3.3. An abelian category $\mathcal{A}$ is called a *Gorenstein category*, if the following conditions are satisfied:
(G1). $\mathcal{A}$ satisfies (AB4) and has enough injective objects, $\mathcal{A}$ has a generator T of finite projective dimension.
(G2). An object X has finite projective dimension if and only if it has finite injective dimension.
(G3). Every injective object has finite projective dimension and these dimensions are uniformly bounded.

Denote by G.dim $\mathcal{A} := \max\{\text{proj.dim } I \mid I \text{ any injective object}\}$, which is called the *Gorenstein dimension* of $\mathcal{A}$.

By the following observation, one deduces that our definition is equivalent to [10, Definition 2.18] in the case of Grothendieck categories.

Proposition 3.4. *Let $\mathcal{A}$ be a Gorenstein category. Then we have*

$$\text{fin.id}\ \mathcal{A} \leq \text{fin.pd}\ \mathcal{A} = \text{G.dim}\ \mathcal{A}.$$

Proof. Assume $X \in \mathcal{A}$ and inj.dim $X = n_0 < \infty$. Then we have an exact sequence

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n_0-1} \xrightarrow{\partial} I^{n_0} \longrightarrow 0,$$

where each I^i is injective and the epimorphism ∂ is not split, in particular, we have $\mathrm{Ext}^1_{\mathcal{A}}(I^{n_0}, \mathrm{Ker}\ \partial) \neq 0$. By the dimension-shift technique in homological algebra, we have $\mathrm{Ext}^{n_0}_{\mathcal{A}}(I^{n_0}, X) \simeq \mathrm{Ext}^1_{\mathcal{A}}(I^{n_0}, \mathrm{Ker}\ \partial) \neq 0$. Therefore, $n_0 \leq$ proj.dim $I^{n_0} \leq$ G.dim $\mathcal{A}$, and thus fin.id $\mathcal{A} \leq$ G.dim $\mathcal{A}$.

By (G2) we have G.dim $\mathcal{A} \leq$ fin.pd $\mathcal{A}$. On the other hand, assume that X has finite projective dimension. By (G2), it has finite injective dimension, and we take its injective resolution $X \longrightarrow I^\bullet$ as above. Hence proj.dim $X =$ proj.dim $I^\bullet$. Applying Lemma 3.1, we get proj.dim $I^\bullet \leq \max\{\text{proj.dim } I^n - n \mid 0 \leq n \leq n_0\}$, in particular, proj.dim $I^\bullet \leq$ G.dim $\mathcal{A}$. From this, one infers that fin.pd $\mathcal{A} \leq$ G.dim $\mathcal{A}$. Then we are done. ■

We observe that the condition (G2) can be characterized in terms of the derived category (compare [15, Lemma 1.5 (iii)]).

Proposition 3.5. *Let $\mathcal{A}$ be an abelian category satisfying (G1). Then the condition (G2) is equivalent to the condition that $D^b(\mathcal{A})_{\mathrm{fpd}} = D^b(\mathcal{A})_{\mathrm{fid}}$.*

Proof. Assume that the condition (G2) holds. First observe that $D^b(\mathcal{A})_{\mathrm{fid}} \subseteq D^b(\mathcal{A})_{\mathrm{fpd}}$. In fact, if $X^\bullet \in D^b(\mathcal{A})_{\mathrm{fid}}$, then by Lemma 3.2, there is a quasi-isomorphism $X^\bullet \longrightarrow I^\bullet$ with $I^\bullet$ a bounded complex of injective objects. By (G2) each I^i has finite projective dimension, and then by Lemma 3.1, we obtain that proj.dim $I^\bullet < \infty$, and thus proj.dim $X^\bullet < \infty$. On the other hand, given $X^\bullet \in D^b(\mathcal{A})_{\mathrm{fpd}}$, by the dual of [16, Chapter I, Lemma 4.6(1)], we may take a quasi-isomorphism $T^\bullet \longrightarrow X^\bullet$, where $T^\bullet = (T^n, \delta^n)$ is a bounded-above complex such that each T^n is a coproduct of copies of T. In particular, proj.dim $T^n \leq$ proj.dim $T < \infty$, and by (G2) inj.dim $T^n < \infty$. Now we will follow the idea in the proof of [23, Appendix, Lemma A.1]. Take $n_0 >> 0$ such that $H^{-n}(T^\bullet) = 0$ for $n \geq n_0$ and $n_0 \geq$ proj.dim $X^\bullet$. Consider the good truncated complex $\tau^{\geq -n_0}T^\bullet := 0 \longrightarrow \mathrm{Coker}\ \delta^{-n_0-2} \longrightarrow T^{-n_0} \longrightarrow T^{-n_0+1} \longrightarrow \cdots$, which is isomorphic to $X^\bullet$ in $D^b(\mathcal{A})$. Then we have the following natural triangle

$$\sigma^{\geq -n_0}T^\bullet \longrightarrow \tau^{\geq -n_0}T^\bullet \longrightarrow \mathrm{Coker}\ \delta^{-n_0-2}[n_0+1] \longrightarrow \sigma^{\geq -n_0}T^\bullet[1].$$

Since proj.dim $\tau^{\geq -n_0}T^\bullet \leq n_0$, the middle morphism in the triangle is zero, and hence the first morphism is split-epi, in other words, $\tau^{\geq -n_0}T^\bullet$ is a direct summand of $\sigma^{\geq -n_0}T^\bullet$. As we noted above that inj.dim $T^n < \infty$ for each n, and then by Lemma 3.1, we have inj.dim $\sigma^{\geq -n_0}T^\bullet < \infty$. So we deduce that inj.dim $\tau^{\geq -n_0}T^\bullet < \infty$, and since $X^\bullet$ is isomorphic to $\tau^{\geq -n_0}T^\bullet$, we obtain that $X^\bullet$ also has finite injective dimension.

Now assume that $D^b(\mathcal{A})_{\mathrm{fpd}} = D^b(\mathcal{A})_{\mathrm{fid}}$. Then (G2) is immediate only if one notes that an object has finite projective dimension (resp. finite injective dimension) if and only if it, as a stalk complex, lies in $D^b(\mathcal{A})_{\mathrm{fpd}}$ (resp. $D^b(\mathcal{A})_{\mathrm{fid}}$). ■

4. Quotients of Gorenstein Categories

In this section, we will consider the question of when a quotient category of a Gorenstein category is still Gorenstein, and we will prove Theorem A in a general form.

4.1. Use the notation in **2.1**. Assume that the abelian category $\mathcal{A}$ satisfies (AB4) and the Serre subcategory $\mathcal{B}$ is closed under arbitrary coproducts, then ones sees that the corresponding multiplicative system Σ is closed under coproducts, and furthermore the quotient category $\mathcal{A}/\mathcal{B}$ has arbitrary coproducts and these coproducts preserve short exact sequences, that is, the quotient category $\mathcal{A}/\mathcal{B}$ also satisfies (AB4). Note that in this case, the functor π preserves coproducts.

Our main result in this section is :

Theorem 4.1. *Let $\mathcal{A}$ be a Gorenstein category with a generator T of finite projective dimension, and $\mathcal{B} \subseteq \mathcal{A}$ a Serre subcategory closed under coproducts, $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$ the canonical functor. Assume that the functor π sends injective objects to injective objects and* $\text{proj.dim}\ \pi(T) < \infty$*. Then the quotient category $\mathcal{A}/\mathcal{B}$ is Gorenstein.*

Moreover, we have $\text{G.dim}\ \mathcal{A}/\mathcal{B} \leq \text{G.dim}\ \mathcal{A} + \text{proj.dim}\ \pi(T)$.

Proof. Let us verify the defining conditions (G1)-(G3) for the quotient category $\mathcal{A}/\mathcal{B}$. For (G1), as we noted above that $\mathcal{A}/\mathcal{B}$ satisfies (AB4); and for each object $\pi(X)$, there is a monomorphism $i : X \longrightarrow I$ in $\mathcal{A}$ with I injective, then by assumption $\pi(I)$ is injective in which $\pi(X)$ embeds, that is, $\mathcal{A}/\mathcal{B}$ has enough injectives; note that every object X is a quotient of a coproduct of copies of T, hence $\pi(X)$ is a quotient of a coproduct of copies of $\pi(T)$ (using that π preserves coproducts), hence $\pi(T)$ is a generator, and by assumption, it is of finite projective dimension.

For the condition (G2), first note that since π is exact and preserves injectives, π sends objects of finite injective dimension to objects of finite injective dimension. In particular, if T' is a coproduct of copies of T, then by (G2) of $\mathcal{A}$, it has finite injective dimension, then we infer that $\pi(T')$ has finite injective dimension. Take $\pi(X) \in \mathcal{A}/\mathcal{B}$. Assume that $\text{proj.dim}\ \pi(X) = n_0 < \infty$. Since $\pi(T)$ is a generator, we may take an exact sequence

$$0 \longrightarrow \pi(K) \longrightarrow \pi(T^{-n_0}) \longrightarrow \pi(T^{-n_0+1}) \longrightarrow \cdots \longrightarrow \pi(T^0) \longrightarrow \pi(X) \longrightarrow 0,$$

where each T^{-i} is a coproduct of copies of T. We have a natural triangle in $D^b(\mathcal{A}/\mathcal{B})$:

$$\mathcal{T}^\bullet \longrightarrow \pi(X) \xrightarrow{\xi} \pi(K)[n_0+1] \longrightarrow \mathcal{T}^\bullet[1],$$

where $\mathcal{T}^\bullet = 0 \longrightarrow \pi(T^{-n_0}) \longrightarrow \pi(T^{-n_0+1}) \longrightarrow \cdots \longrightarrow \pi(T^0) \longrightarrow 0$. Since $\text{proj.dim}\ \pi(X) = n_0$, we infer that ξ is zero, that is, the leftmost morphism is split-epi. Hence $\pi(X)$ is a direct summand of $\mathcal{T}^\bullet$, hence $\text{inj.dim}\ \pi(X) \leq \text{inj.dim}\ \mathcal{T}^\bullet$. Here again we have used the idea in the proof of [23, Appendix, Lemma A.1]. As we noted above, each $\pi(T^{-i})$ has finite injective dimension, hence by Lemma 3.1, $\text{inj.dim}\ \mathcal{T}^\bullet < \infty$. Therefore $\pi(X)$ has finite injective dimension.

On the other hand, assume that $\text{inj.dim}\ \pi(X) = n_0 < \infty$. We will show that $\text{proj.dim}\ \pi(X) < \infty$. Then we have an exact sequence $0 \longrightarrow \pi(X) \longrightarrow \mathcal{I}^0 \longrightarrow \cdots \longrightarrow \mathcal{I}^{n_0-1} \longrightarrow \mathcal{I}^{n_0} \longrightarrow 0$ with each $\mathcal{I}^i$ injective. By Lemma 3.1 it suffices to show that each $\mathcal{I}^i$, or more generally, any injective object $\mathcal{I}$ in $\mathcal{A}/\mathcal{B}$ has finite projective dimension. We claim that $\mathcal{I}$ is direct summand of $\pi(I)$ for some injective object I in $\mathcal{A}$. In fact, take $X' \in \mathcal{A}$ such that $\pi(X') = \mathcal{I}$, embed X' into an injective object I, thus $\mathcal{I}$ embeds into $\pi(I)$, and this embedding is necessarily split since $\mathcal{I}$ is injective, and this proves the claim. So to prove (G2), it suffices to show that for each injective I in $\mathcal{A}$, $\text{proj.dim}\ \pi(I) < \infty$. By (G3) of $\mathcal{A}$, assume that $\text{proj.dim}\ I = d \leq \text{G.dim}\ \mathcal{A}$. Then by a similar argument as above, we have that in $D^b(\mathcal{A})$, I is a direct summand of a complex $T^\bullet = 0 \longrightarrow T^{-d} \longrightarrow T^{-d+1} \longrightarrow \cdots \longrightarrow T^0 \longrightarrow 0$ with each T^{-i}

a coproduct of copies of T. Hence applying π, we have that in $D^b(\mathcal{A}/\mathcal{B})$, $\pi(I)$ is a direct summand of $\pi(T^\bullet)$, and thus proj.dim $\pi(I) \leq$ proj.dim $\pi(T^\bullet)$. Since by assumption $\pi(T)$ has finite projective dimension, so does any coproducts of its copies, more precisely, proj.dim $\pi(T^{-i}) \leq$ proj.dim $\pi(T)$. Applying Lemma 3.1, we obtain that proj.dim $\pi(T^\bullet) \leq d +$ proj.dim $\pi(T)$. Consequently, we have proj.dim $\pi(I) \leq$ G.dim $\mathcal{A}$ + proj.dim $\pi(T)$. This finishes the proof of (G2) and this also proves (G3), and even more, one infers that G.dim $\mathcal{A}/\mathcal{B} \leq$ G.dim $\mathcal{A}$ + proj.dim $\pi(T)$. We are done. ■

4.2. In this section, we will apply Theorem 4.1 to the situation where the abelian category $\mathcal{A}$ is Grothendieck and $\mathcal{B} \subseteq \mathcal{A}$ is a localizing Serre subcategory.

Before fixing our setup we recall some notions. Recall that a monomorphism $i : X \longrightarrow Y$ is *essential* if for any morphism $g : Y \longrightarrow Z$, the composite $g \circ i$ is mono implies that g is mono; the *injective hull* of X means an essential monomorphism $i : X \longrightarrow I$ with I injective. We say a full subcategory $\mathcal{B}$ is *closed under essential monomorphisms*, if for any essential monomorphism $i : X \longrightarrow Y$ with $X \in \mathcal{B}$, then Y lies also in $\mathcal{B}$. Recall that an abelian category $\mathcal{A}$ is said to be *a Grothendieck category* if it satisfies (AB4), direct limits in $\mathcal{A}$ are exact, and it has a generator. Note that in a Grothendieck category, any object has a (unique) injective hull ([26, Chapter V]). Assume that $\mathcal{A}$ is Grothendieck and $\mathcal{B} \subseteq \mathcal{A}$ is a localizing Serre subcategory, then the quotient category $\mathcal{A}/\mathcal{B}$ is also Grothendieck (for details, see [26, Chapter X])

Setup 4.2: The abelian category $\mathcal{A}$ is Grothendieck, and $\mathcal{B} \subseteq \mathcal{A}$ is a localizing Serre subcategory which is closed under essential monomorphisms.

Let us draw some immediate consequences of this setup.

Lemma 4.2. *Assume Setup 4.2. Then we have*
(1). Both the functors inc $: \mathcal{B} \longrightarrow \mathcal{A}$ *and* $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$ *preserve injective objects.*
(2). The subcategory $\mathcal{B}$ has enough $\mathcal{A}$-injective objects.

Proof. Let I be any injective object in $\mathcal{A}$. We first claim that its subobject $\tau(I)$ is injective. In fact, consider the injective hull $i : \tau(I) \longrightarrow I'$. Since i is an essential monomorphism, by the assumption we get that $I' \in \mathcal{B}$. However by the injectiveness of I', there is a morphism $i' : I' \longrightarrow I$ extending the inclusion of $\tau(I)$ into I, which is necessarily a monomorphism since $i' \circ i$ is mono and i is essential. Note that $\tau(I)$ is the unique largest subobject of I belonging to $\mathcal{B}$, hence the image of i' lies in $\tau(I)$, and consequently we deduce that i is an isomorphism, and therefore $\tau(I)$ is injective.

Assume that I is injective as above. Then $\tau(I)$ is also injective hence it is direct summand of I, say $I \simeq \tau(I) \oplus I'$. Then we have that I' is injective and $\tau(I') = 0$, and thus I' is $\mathcal{B}$-local, by [24, Lemma A. 2.7], we have a natural isomorphism $\mathrm{Hom}_{\mathcal{A}}(X, I') \simeq \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(\pi(X), \pi(I'))$ for any $X \in \mathcal{A}$, and note that any short exact sequence in $\mathcal{A}/\mathcal{B}$ is isomorphic to the image of some short exact sequence in $\mathcal{A}$, we infer that the functor $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(-, \pi(I'))$ is exact, and thus $\pi(I')$ is injective. Note that $\pi(I) \simeq \pi(I')$, we get that π preserves injective objects.

Let $B \in \mathcal{B}$, take its injective hull in $\mathcal{A}$, $i : B \longrightarrow I$. Since $\mathcal{B}$ is closed under essential monomorphisms, we have that I belongs to $\mathcal{B}$, and thus this proves (2). Moreover if B is $\mathcal{B}$-injective, then $i : B \longrightarrow I$ splits and thus B is a direct summand of I, so we have that B is also injective in $\mathcal{A}$, in other words, the inclusion functor inc preserves injective objects. We are done. ■

Recall that the *cohomological dimension* ([16, p.57]) of the functor $\omega : \mathcal{A}/\mathcal{B} \longrightarrow \mathcal{A}$ is defined to be

$$\mathrm{coh.dim}\, \omega := \sup \{n \geq 0 \mid \text{the } n\text{-th right derived functor } R^n\omega \neq 0\}.$$

Similarly one defines the cohomological dimension of τ, denoted by coh.dim τ.

Proposition 4.3. *Assume Setup 4.2. Assume that $\mathcal{A}$ has a generator T of finite projective dimension. Then* $\mathrm{proj.dim}\, \pi(T) < \infty$ *if and only if* $\mathrm{coh.dim}\, \omega < \infty$*, if and only if* $\mathrm{coh.dim}\, \tau < \infty$.

Moreover, we have $\mathrm{coh.dim}\ \omega = \max\ \{0, \mathrm{coh.dim}\ \tau - 1\}$ *and* $\mathrm{coh.dim}\ \omega \leq \mathrm{proj.dim}\ \pi(T) \leq \mathrm{coh.dim}\, \omega + \mathrm{proj.dim}\, T$.

Proof. By Lemma 4.2, we can apply Lemma 2.1 in this case. By the right recollement in the lemma, we have for each object $X \in \mathcal{A}$, a triangle in the derived category $R^+\tau(X) \longrightarrow X \longrightarrow R^+\omega(\pi(X)) \longrightarrow R^+\tau(X)[1]$, and taking their cohomologies, we infer that $R^{i+1}\tau(X) \simeq R^i\omega(\pi(X))$ for each $i \geq 1$. Thus it follows that coh.dim $\omega < \infty$ if and only if coh.dim $\tau < \infty$, and moreover, we have $\mathrm{coh.dim}\, \omega = \max\, \{0, \mathrm{coh.dim}\, \tau - 1\}$ (compare [23, p.521, line 13]).

To continue the proof, let us recall two facts: let $I^\bullet = (I^n, d^n)_{n\in\mathbb{Z}}$ be a complex of injective objects in $\mathcal{A}$. The first fact is, for any fixed n, if $H^n(I^\bullet) \neq 0$, then $H^n(\mathrm{Hom}_\mathcal{A}(T, I^\bullet)) \neq 0$. In fact, in this case, the natural inclusion $\mathrm{Im}\ d^{n-1} \longrightarrow \mathrm{Ker}\ d^n$ is proper, thus since T is a generator, there exists $f : T \longrightarrow \mathrm{Ker}\ d^n$ which does not factor through $\mathrm{Im}\ d^{n-1}$. Then one observes that this f gives a nonzero element in the cohomology group $H^n(\mathrm{Hom}_\mathcal{A}(T, I^\bullet))$. The second is that, if there exists n_0 such that $H^n(I^\bullet) = 0$ whenever $n \geq n_0$, then $H^n(\mathrm{Hom}_\mathcal{A}(T, I^\bullet)) = 0$ for all $n \geq n_0 + \mathrm{proj.dim}\, T$. Note that the brutal truncated complex $\sigma^{\geq n_0-1}I^\bullet$ is a shifted version of an injective resolution of $\mathrm{Ker}\ d^{n_0-1}$, and thus we have $H^n(\mathrm{Hom}_\mathcal{A}(T, I^\bullet)) \simeq \mathrm{Ext}^{n-n_0+1}_\mathcal{A}(T, \mathrm{Ker}\ d^{n_0-1})$, and thus the fact follows.

Take an arbitrary object $\mathcal{M} \in \mathcal{A}/\mathcal{B}$, and take its injective resolution $\mathcal{M} \longrightarrow \mathcal{I}^\bullet$. Then we have natural isomorphisms, for all $n \geq 0$,

$$\mathrm{Ext}^n_{\mathcal{A}/\mathcal{B}}(\pi(T), \mathcal{M}) = H^n(\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(\pi(T), \mathcal{I}^\bullet)) \simeq H^n(\mathrm{Hom}_\mathcal{A}(T, \omega(\mathcal{I}^\bullet))).$$

Set $I^\bullet = \omega(\mathcal{I}^\bullet)$, which is a bounded-below complex of injective objects by Lemma 4.2 (1). Note that $H^i(I^\bullet) = R^i\omega(\mathcal{M})$. Then one deduces the result from the above recalled two facts immediately. ∎

The following is a direct consequence of Theorem 4.1.

Corollary 4.4. *Assume Setup 4.2. Suppose furthur that the category $\mathcal{A}$ is Gorenstein and the torsion functor τ has finite cohomological dimension. Then the quotient category $\mathcal{A}/\mathcal{B}$ Gorenstein.*

Proof. Just note that localizing subcategory is always closed under coproducts. Now by Lemma 4.2 (1) and Proposition 4.3, we may apply Theorem 4.1, and thus we are done. ∎

5. A Right Recollement for Gorenstein-Injective Objects

In this section, we will give a right recollement of the stable category of Gorenstein-injective objects under the same conditions of Corollary 4.4, proving Theorem B in a general form.

5.1. We will recall the concept of Gorenstein-injective objects. Let $\mathcal{A}$ be an abelian category with enough injective objects for this moment. Recall that a complex $I^\bullet$ of injective objects is said to be *totally-acyclic*, if it is exact (= acyclic) and for each injective object Q, the Hom complex $\mathrm{Hom}_\mathcal{A}(Q, I^\bullet)$ is exact. An object $G \in \mathcal{A}$ is said to be *Gorenstein-injective* provided there exists a totally-acyclic complex $I^\bullet$ such that $G = Z^0(I^\bullet)$ is the 0-th cocycle, and in this case, $I^\bullet$ is said to be a *complete resolution* of G. Denote by $\mathrm{GInj}(\mathcal{A})$ the full subcategory of Gorenstein-injective objects ([11, Chapter 10] and [21, section 7]). Note that this is an additive subcategory and injective objects are Gorenstein-injective. Observe that for each Gorenstein-injective object G, we have $\mathrm{Ext}^i_\mathcal{A}(Q, G) = 0$ for any injective object Q and $i \geq 1$. (In fact, view the bruntal truncated complex $\sigma^{\geq 0} I^\bullet$ as an injective resolution of G, and then we see that $\mathrm{Ext}^i_\mathcal{A}(Q, G)$ is just the i-th cohomology group of the Hom complex $\mathrm{Hom}_\mathcal{A}(Q, I^\bullet)$, which is zero by the assumption.)

We collect the basic properties of the category $\mathrm{GInj}(\mathcal{A})$.

Lemma 5.1. *With the notation as above. We have*
(1). The full subcategory $\mathrm{GInj}(\mathcal{A})$ *is closed under cokernels of monomorphisms, extensions and taking direct summands.*
(2). Endow the exact structure on $\mathrm{GInj}(\mathcal{A})$ *by short exact sequences in* $\mathcal{A}$*. Then* $\mathrm{GInj}(\mathcal{A})$ *is a Frobenius category in the sense of* [14] *(and* [18, p.381]*), whose relative injective-projective objects are precisely the injective objects in* $\mathcal{A}$*.*

Proof. The first statement is proved in [11, Theorem 10.1.4] and also by the dual of [2, Proposition 5.1]. Since $\mathrm{GInj}(\mathcal{A})$ is closed under extensions, then it becomes an exact category in the sense of Quillen by identifying conflations (= admissible exact sequences) with short exact sequences in $\mathcal{A}$ with terms inside $\mathrm{GInj}(\mathcal{A})$, see [18, Appendix A]. Clearly injective objects are relative injective and thus $\mathrm{GInj}(\mathcal{A})$ has enough relative injectives. Note that $\mathrm{Ext}^1_\mathcal{A}(Q, G) = 0$ for injective objects Q and $G \in \mathrm{GInj}(\mathcal{A})$, and this proves that the functor $\mathrm{Hom}_\mathcal{A}(Q, -)$ is exact on all the conflations, i.e., Q is a relative projective. Observe that for any $G \in \mathrm{GInj}(\mathcal{A})$, there exists an exact sequence $0 \longrightarrow G' \longrightarrow Q \longrightarrow G \longrightarrow 0$ with G' Gorenstein-injective and Q injective in $\mathcal{A}$, and since as we just saw, Q is relative projective, therefore $\mathrm{GInj}(\mathcal{A})$ has enough relative projectives. Now it is not hard to see (2). ■

The following important result could be derived from [10, Theorem 2.24].

Proposition 5.2. *Let $\mathcal{A}$ be a Grothendick category which is Gorenstein, $M \in \mathcal{A}$ an object. Then the following are equivalent:*
(1). The object M is Gorenstein-injective.
(2). There exists a long exact sequence $\cdots \longrightarrow I^{-n} \longrightarrow \cdots \longrightarrow I^{-1} \longrightarrow I^0 \longrightarrow M \longrightarrow 0$ *with each* I^{-n} *injective.*
(3). For each injective Q and $i \geq 1$, $\mathrm{Ext}^i_\mathcal{A}(Q, M) = 0$*.*
(4). For each object L of finite injective dimension, $\mathrm{Ext}^1_\mathcal{A}(L, M) = 0$*.*

Proof. The implication "(1) $\Longrightarrow$ (2)" is trivial from the definition. To see "(2) $\Longrightarrow$ (3)", we apply the dimension-shift technique in homological algebra and then we get, for any $i \geq 1$ and $k \geq 0$, $\mathrm{Ext}^i_\mathcal{A}(Q, M) \simeq \mathrm{Ext}^{i+k+1}_\mathcal{A}(Q, Z^k)$, where Z^k is the $-k$-th cocycle of the long exact complex. Note that by (G2), Q has finite projective dimension, and we infer that $\mathrm{Ext}^i_\mathcal{A}(Q, M) = 0$. "(3) $\Longrightarrow$ (4)" could also be shown by

the dimension-shift technique. While "(4) $\Longrightarrow$ (1)" is exactly stated in [10, Theorem 2.24]. ■

5.2. Let $\mathcal{A}$ be an abelian category with enough injectives and let $\mathrm{GInj}(\mathcal{A})$ be its subcategory of Gorenstein-injective objects. Consider the stable category $\underline{\mathrm{GInj}}(\mathcal{A})$ modulo injective objects: the objects are the same as in $\mathrm{GInj}(\mathcal{A})$, and the morphism space is the quotient of the corresponding one modulo those factoring through injective objects. For each morphism $f : X \longrightarrow Y$, we write the corresponding morphism in $\underline{\mathrm{GInj}}(\mathcal{A})$ as $\underline{f} : X \longrightarrow Y$. Since $\mathrm{GInj}(\mathcal{A})$ is a Frobenius category, then by [14, Chapter 1, section 2], the stable category $\underline{\mathrm{GInj}}(\mathcal{A})$ has a canonical triangulated structure as follows: for each $X \in \mathrm{GInj}(\mathcal{A})$, fix a short exact sequence $0 \longrightarrow X \xrightarrow{i_X} I(X) \xrightarrow{d_X} S(X) \longrightarrow 0$ such that $I(X)$ is injective and thus $S(X)$ belongs to $\mathrm{GInj}(\mathcal{A})$, and thus we have the induced functor $S : \underline{\mathrm{GInj}}(\mathcal{A}) \longrightarrow \underline{\mathrm{GInj}}(\mathcal{A})$ which is an auto-equivalence and will be the shift functor; triangles in $\underline{\mathrm{GInj}}(\mathcal{A})$ are induced by conflations, i.e., the short exact sequences in $\mathrm{GInj}(\mathcal{A})$, more precisely, given a short exact sequence $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$, we have a commutative diagram in $\mathcal{A}$ (using the relative injectivity of $I(X)$)

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, dotted] & Z \arrow[r] \arrow[d, dotted, "w"] & 0 \\
0 \arrow[r] & X \arrow[r, "i_X"] & I(X) \arrow[r, "d_X"] & S(X) \arrow[r] & 0.
\end{tikzcd}$$

Then $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{-\underline{w}} S(X)$ is a triangle, and all triangles arise in this way ([8, Lemma 1.2]).

Our main observation is that under the conditions of Corollary 4.4, we have a natural right recollement of triangulated categories relating the Gorenstein-injective objects of the abelian category and of its quotient category.

Theorem 5.3. *Assume Setup 4.2. Assume that $\mathcal{A}$ is Gorenstein and the torsion functor τ has finite cohomological dimension. Then we have a right recollement of triangulated categories:*

$$\underline{\mathrm{GInj}}(\mathcal{B}) \underset{\underline{\tau}}{\overset{\mathrm{inc}}{\rightleftarrows}} \underline{\mathrm{GInj}}(\mathcal{A}) \underset{\underline{\omega}}{\overset{\underline{\pi}}{\rightleftarrows}} \underline{\mathrm{GInj}}(\mathcal{A}/\mathcal{B}),$$

where the functors are induced from the ones between the abelian categories.

Proof. We will first show that the four functors involved are well-defined. This will take several steps.

We claim that for each $i \geq 1$ and $X \in \mathrm{GInj}(\mathcal{A})$, $R^i\tau(X) = 0$. Consequently, τ preserves short exact sequences in $\mathrm{GInj}(\mathcal{A})$. In fact, since X is Gorenstein-injective, we have an exact sequence $\cdots \longrightarrow I^{-n} \xrightarrow{d^{-n}} \cdots \longrightarrow I^{-1} \xrightarrow{d^{-1}} I^0 \longrightarrow X \longrightarrow 0$ with each I^{-n} injective. Thus by the dimension-shift technique, we get $R^i\tau(X) \simeq R^{i+k}\tau(\mathrm{Im}\ d^{-k})$ for any $k \geq 1$. However τ has finite cohomological dimension, thus we are done with $R^i\tau(X) = 0$, $i \geq 1$. The other statement follows directly from the long exact sequence associated with a given short exact sequence and the derived functors $R^i\tau$.

Next we claim that $\mathrm{GInj}(\mathcal{B}) \subseteq \mathrm{GInj}(\mathcal{A}) \cap \mathcal{B}$. To see this, let $X \in \mathrm{GInj}(\mathcal{B})$, and then from the definition of Gorenstein-injective objects, we have an exact sequence

$\cdots \longrightarrow I^{-n} \longrightarrow \cdots \longrightarrow I^{-1} \longrightarrow I^0 \longrightarrow X \longrightarrow 0$ with each I^{-n} injective in $\mathcal{B}$, however as we observed in Lemma 4.2 (1), each I^{-n} is also injective in $\mathcal{A}$, and since $\mathcal{A}$ is Gorenstein and by Proposition 5.2 (2), X belongs to $\mathrm{GInj}(\mathcal{A})$.

We claim that if $X \in \mathrm{GInj}(\mathcal{A})$, then $\tau(X) \in \mathrm{GInj}(\mathcal{B})$. Consequently, combining above we infer that $\mathrm{GInj}(\mathcal{B}) = \mathrm{GInj}(\mathcal{A}) \cap \mathcal{B}$. Take the complete resolution $I^\bullet = (I^n, d^n)$ with $Z^0(I^\bullet) = X$. Break $I^\bullet$ into short exact sequences in $\mathrm{GInj}(\mathcal{A})$ and note that by the first claim τ preserves such short exact sequences, we obtain that the complex $\tau(I^\bullet)$ is exact and $Z^0(\tau(I^\bullet)) = \tau(X)$. Note that τ preserves injective objects (which is a consequence of a general fact that the right adjoint functor of an exact functor preserves injectives), hence $\tau(I^\bullet)$ is an exact sequence of injective objects in $\mathcal{B}$. Given any injective Q in $\mathcal{B}$, then by Lemma 4.2 (1), Q is also injective in $\mathcal{A}$. Note that by adjoint we have an isomorphism of Hom complexes $\mathrm{Hom}_\mathcal{B}(Q, \tau(I^\bullet)) \simeq \mathrm{Hom}_\mathcal{A}(Q, I^\bullet)$, and since $I^\bullet$ is totally-acyclic, and hence $\mathrm{Hom}_\mathcal{A}(Q, I^\bullet)$ is cyclic , and so $\mathrm{Hom}_\mathcal{B}(Q, \tau(I^\bullet))$ is also acyclic, in other words, $\tau(I^\bullet)$ is a totally-acyclic complex in $\mathcal{B}$ and note that $Z^0(\tau(I^\bullet)) = \tau(X)$, therefore $\tau(X) \in \mathrm{GInj}(\mathcal{B})$.

Let us summarize what we have shown: we have two well-defined functors inc : $\mathrm{GInj}(\mathcal{B}) \longrightarrow \mathrm{GInj}(\mathcal{A})$ and $\tau : \mathrm{GInj}(\mathcal{A}) \longrightarrow \mathrm{GInj}(\mathcal{B})$, both of which preserve relative injective-projective objects and short exact sequences. Therefore we have two induced functors on the stable categories and they are triangle functors by [14, p.23, Lemma]. Obviously the obtained pair $(\mathrm{inc}, \underline{\tau})$ is adjoint and the functor inc is fully-faithful.

To continue, we claim that if $X \in \mathrm{GInj}(\mathcal{A})$, then $\pi(X) \in \mathrm{GInj}(\mathcal{A}/\mathcal{B})$. Take the complete resolution $I^\bullet$ for X, then by Lemma 4.2(1) $\pi(I^\bullet)$ is an exact sequence of injective objects and its 0-th cocycle is $\pi(X)$. By Corollary 4.4, $\mathcal{A}/\mathcal{B}$ is Gorenstein. Applying Proposition 5.2(2), we get that $\pi(X)$ is Gorenstein-injective.

We claim that for each $i \geq 1$ and $\mathcal{M} \in \mathrm{GInj}(\mathcal{A}/\mathcal{B})$, $R^i\omega(\mathcal{M}) = 0$. Consequently, ω preserves short exact sequences in $\mathrm{GInj}(\mathcal{A}/\mathcal{B})$. Note that by Proposition 4.3, the functor ω has also finite cohomological dimension, and the proof is similar as the first claim, and left to the reader.

We claim that if $\mathcal{M} \in \mathrm{GInj}(\mathcal{A}/\mathcal{B})$, then $\omega(\mathcal{M}) \in \mathrm{GInj}(\mathcal{A})$. Note that by Lemma 4.2 (1), π preserves injective objects, and using the last claim, we can prove this by an argument similar to that in the third claim. And so the proof is left to the reader.

Hence we have obtained two well-defined functors: $\pi : \mathrm{GInj}(\mathcal{A}) \longrightarrow \mathrm{GInj}(\mathcal{A}/\mathcal{B})$ and $\omega : \mathrm{GInj}(\mathcal{A}/\mathcal{B}) \longrightarrow \mathrm{GInj}(\mathcal{A})$, both of which preserve relative injective-projective objects and short exact sequences. Therefore we get two induced triangle functors $\underline{\pi}$ and $\underline{\omega}$ on the stable categories, and the pair $(\underline{\pi}, \underline{\omega})$ is adjoint. Note that the functor $\underline{\omega}$ is obviously fully-faithful and the composite $\underline{\pi}\ \mathrm{inc} = 0$.

To end the proof, it suffices to check the last defining condition of a right recollement. In fact, from the right recollement in Lemma 2.1, we get for each $X \in \mathcal{A}$, a triangle in the derived category $R^+\tau(X) \longrightarrow X \longrightarrow R^+\omega(\pi(X)) \longrightarrow R^+\tau(X)[1]$. From the triangle we get a long exact sequence of cohomological groups

$$0 \longrightarrow \tau(X) \longrightarrow X \longrightarrow \omega(\pi(X)) \longrightarrow R^1\tau(X) \longrightarrow \cdots$$

Take $X \in \mathrm{GInj}(\mathcal{A})$ and by the first claim, we get an exact sequence in $\mathcal{A}$, even in $\mathrm{GInj}(\mathcal{A})$, $0 \longrightarrow \tau(X) \longrightarrow X \longrightarrow \omega(\pi(X)) \longrightarrow 0$, and note that the morphisms involved are the adjunction morphisms. Short exact sequences induce triangles, and thus we get the required triangle for the last defining condition of a right recollement. Then we are done. ■

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